Goldbach Conjecture for Laurent Polynomials with Positive Coefficients

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(Joint work with H. Polo)

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Overview

1. Terminology
2. Motivation
3. Irreducibility Criteria
4. Proof
   - Monic
   - General

Terminology

- Laurent polynomials $\to$ elements of $\mathbb{N}_0[x^{\pm 1}]$.
  - Ex: $x^3 + 2x^{-2}$.
- A polynomial f's size $\to f(1)$.
- Let $f = \sum_{i=0}^{n} c_i x^{k_i}$ where $c_i > 0$.

  We say $\mathrm{supp}(f) = \{k_i \mid i \in [\![0, n]\!]\}$.
- An element a is an additive atom if $a = b + c$ implies either $b = 0$ or $c = 0$.

Motivation

$\mathbb{Z}[x]$ $\qquad$ $\mathbb{F}_q[x]$ $\qquad$ $D[x]$

Motivation

Example in $\mathbb{Z}[x]$:

$$
\begin{tikzcd}
& x^2 + x + 1 \arrow[dl] \arrow[dr] & \\
x^2 + 1 & & x \\
x^2 + 2 & & x - 1 \\
x^2 + 3 & & x - 2 \\
\vdots & & \vdots
\end{tikzcd}
$$

Motivation

Example for regular Goldbach conjecture:

$$8$$

$$\downarrow$$

$$\underbrace{1+1+1+1+1+1+1+1}_{8}$$

$$\swarrow \qquad\qquad \searrow$$

$$\underbrace{1+1+1}_{3} \qquad\qquad \underbrace{1+1+1+1+1}_{5}$$

Motivation

What about...

$$\cancel{\mathbb{N}_0[x]?}$$

Counterexample: $x^5 + x^4 + x^3 + x^2$

Instead, we use $\mathbb{N}_0[x^{\pm 1}]$.

Notice that $\mathbb{N}_0[x^{\pm 1}]^{\times} = \{x^k \mid k \in \mathbb{Z}\}$.

Motivation

Theorem

Every polynomial $f \in \mathbb{N}_0[x^{\pm 1}]$ *can be written as the sum of two irreducible elements provided that*

- $f(1) > 3$,
- $|supp(f)| > 1$.

These conditions are necessary:

- Counterexample: 3
- Not necessarily true for odd numbers (*ex:* $11x$) AND we would have to prove the Goldbach conjecture...

Motivation

Why is this an important question?

- There has been interest in similar problems.
- Natural continuation of previous work.
- $\mathbb{N}_0[x^{\pm 1}]$ is a semidomain.
- Working on difficult problems creates new mathematical techniques (*ex: Hardy-Littlewood circle method*).
- It is just cool!

Irreducibility Criteria

Irreducibility criteria:

Let $f \in \mathbb{N}_0[x^{\pm 1}]$.

1. **Prime Criterion.** If

$$f(1) = p, \textit{ for some prime } p,$$

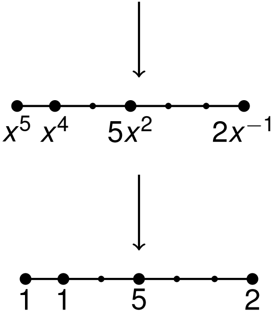
   then f is irreducible.

Irreducibility Criteria

Irreducibility criteria:

Different notation.

$$f = x^5 + x^4 + 5x^2 + 2x^{-1}$$



Irreducibility Criteria

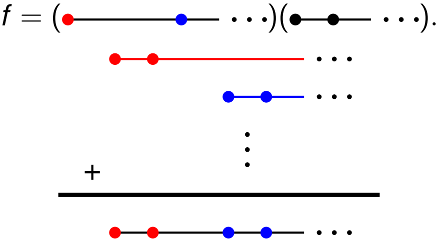
Irreducibility criteria:

Let $f \in \mathbb{N}_0[x^{\pm 1}]$.

2. **Small-Gap Criterion.** If
   (i) *Either leftmost gap of f or the rightmost gap of f is strictly the smallest gap;*
   (ii) *The gcd of the coefficients of f is 1;*

   then f is irreducible.

Suppose f's leftmost gap is strictly the smallest, and suppose



Proof

Theorem

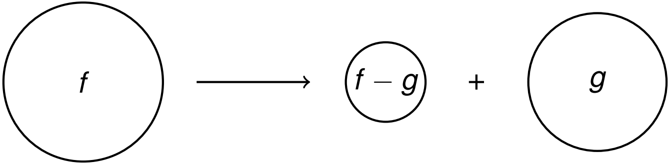
Every monic polynomial $f \in \mathbb{N}_0[x^{\pm 1}]$ can be written as the sum of two irreducible elements provided that

- $f(1) > 3$,
- $|supp(f)| > 1$.

Proof

Proof overview:

- Find a summand g of f such that $\frac{f(1)}{2} \leq g(1)$ and g satisfies the Small-Gap Criterion.



- "Move" additive atoms from g until $(f-g)(1) = p$, for some prime p while maintaining that g satisfies the Small-Gap Criterion.

Proof

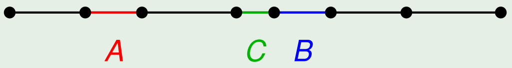
Current step: Find a summand g of f such that $\frac{\mathbf{f(1)}}{\mathbf{2}} \leq \mathbf{g(1)}$ and g satisfies the Small-Gap Criterion.

Lemma

There exist gaps A, B in f such that

- *Gap A is to the left of gap B,*
- *All gaps to the left of A and to the right of B are longer than both A and B,*
- *All gaps between A and B are at least as long as A and B, with at most one exception.*

Example

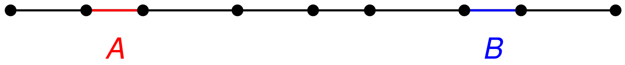
$f =$ 

Proof

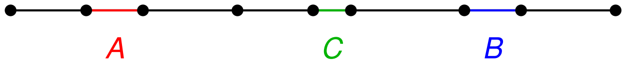
Current step: Find a summand g of f such that $\frac{\mathbf{f(1)}}{\mathbf{2}} \leq \mathbf{g(1)}$ and g satisfies the Small-Gap Criterion.

Suppose the smallest gap has length k.

1. There is more than one gap with length k.



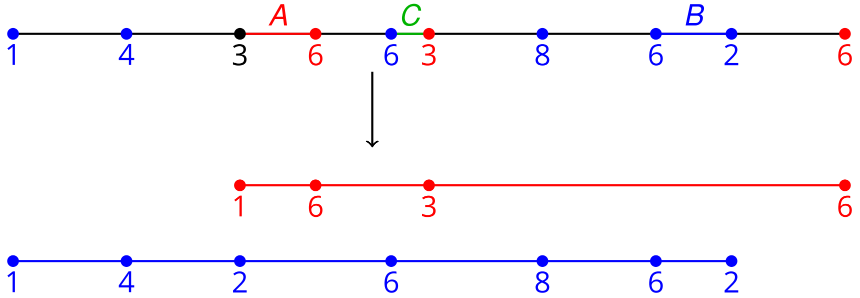
2. There is only one gap with length k.



Proof

Current step: Find a summand g of f such that $\frac{f(1)}{2} \leq g(1)$ and g satisfies the Small-Gap Criterion.

Example. Let
$f = x^{22} + 4x^{19} + 3x^{16} + 6x^{14} + 6x^{12} + 3x^{11} + 8x^8 + 6x^5 + 2x^3 + 6$.

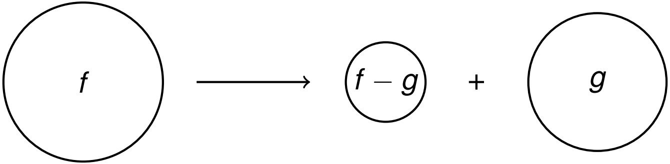


$f - g = 3x^{16} + 6x^{14} + 3x^{11} + 6$
$g = x^{22} + 4x^{19} + 6x^{12} + 8x^8 + 6x^5 + 2x^3$

Proof

Proof overview:

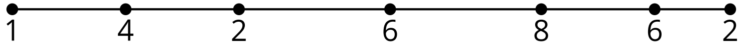
- Find a summand g of f such that $\frac{f(1)}{2} \leq g(1)$ and g satisfies the Small-Gap Criterion.

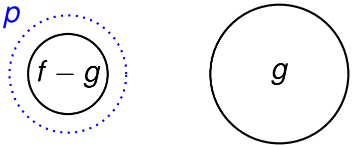


- "Move" additive atoms from g until $(f - g)(1) = p$, for some prime p while maintaining that g satisfies the Small-Gap Criterion.
  - Show that we can subtract up to $g(1) - 2$ additive atoms from g maintaining that g satisfies the Small-Gap Criterion.
  - Find a prime p such that $\frac{f(1)}{2} < p \leq f(1) - 2$.

Proof

Current step: Show that we can subtract up to g$(1) - $2 additive atoms from g maintaining that g satisfies the Small-Gap Criterion.





Proof

Current step: Find a prime p such that $\frac{\mathbf{f(1)}}{\mathbf{2}} < \mathbf{p} \leq \mathbf{f(1) - 2}$.

Let p be the largest prime less than $f(1)$.

Theorem (Bertrand's Postulate)

For any integer $n > 1$, there always exists at least one prime number q with $n < q < 2n$.

So, $p < f(1) < 2p \longrightarrow \frac{f(1)}{2} < p < f(1)$.

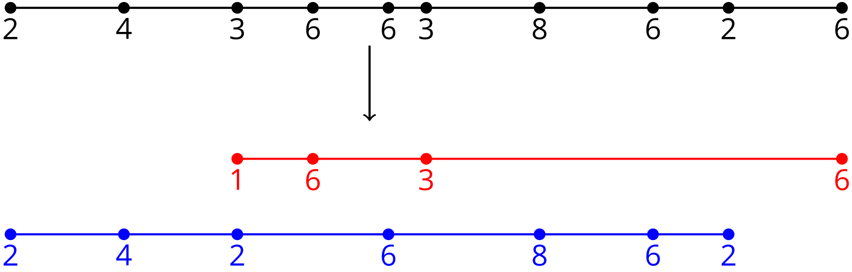
If $p = f(1) - 1$, let p' be the largest prime less than p.

$p' < p < 2p' \longrightarrow \frac{p}{2} < p' < p \longrightarrow \frac{f(1)-1}{2} < p' < f(1) - 1$.

Either $\frac{f(1)}{2} < p' \leq f(1) - 2$ OR $p' = \frac{f(1)}{2}$.

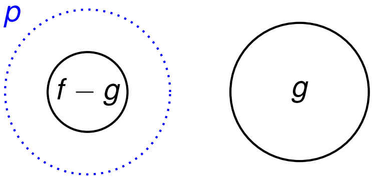
Proof — Generalizing

What if the polynomial was not monic?



Proof — Generalizing

Solution: Make p BIGGER!



Let c be the smallest coefficient of g. When $|\text{supp}(g)| \geq 3$,

$$c \leq \frac{g(1)}{3} \leq \frac{f(1)}{3}.$$

If only $\frac{5}{6}f(1) \leq p$...

Proof — Generalizing

Theorem (Modified Bertrand's Postulate — J. Nagura)

For any integer $n \geq 25$, there always exists at least one prime number q with $n < q < \frac{6}{5}n$.

Conclusion

...To see all the details, read our paper!

Moving Forward

- Goldbach conjecture holds in additively reduced semifields.
- In what types of semidomains does Goldbach conjecture hold?
- What other types of irreducibility criteria can we find for $\mathbb{N}_0[x^{\pm 1}]$?

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Thank you!

Questions?